

On vosperian and superconnected vertex-transitive digraphs

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Abstract

We investigate the structure of a digraph having a transitive automorphism group where every cutset of minimal cardinality consists of all successors or all predecessors of some vertex. We improve most of the existing results in this area.

1 Introduction

By a digraph, we shall mean a finite directed graph having no loops. Let $\Gamma = (V, E)$ be a digraph. The set successors (resp. antecedents) of a vertex $x \in V$ will be denoted by $\Gamma(x)$ ($\Gamma^-(x)$). The degree of x is defined as $|\Gamma(x)|$. If all the vertices have the same degree, the graph will be called regular and this common value will be called the degree of Γ . A subset C of V is said to be *strongly connected* if, for any two vertices x, y of C there is a directed path of Γ from x to y with all vertices contained in C . The digraph Γ is said to be *strongly connected* if V is strongly connected. A subset T of V will be called a *cutset* of Γ if $V \setminus T$ is not strongly connected. The minimal cardinality of a cutset is called the *connectivity* of Γ .

Regular digraphs may have small connectivity. But vertex transitive digraphs have a large one. It was proved independently by Mader [14] and Watkins [18] that the connectivity of a strongly connected vertex-transitive symmetric digraph with degree d is greater than $2d/3$. One of the authors proved in [6] that a strongly connected vertex-transitive digraph with degree d has connectivity greater $d/2$. It was proved by Watkins in [18] that the connectivity of a strongly connected edge-transitive graph is its minimal degree. One of the authors proved in [6] that the connectivity of a strongly connected arc-transitive digraphs is its degree.

A digraph Γ with degree d is said to be *superconnected* if every cutset with cardinality less than $d + 1$ consists of all successors or all predecessors of some vertex. This notion, introduced in the undirected case, by Boesch and Tindell [3], was also investigated among others by Fiol [5] and Balbuena and Carmona [1]. For an introduction to superconnectivity and related topics, we recommend the chapter by Fàbrega and Fiol in [4].

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If any cutset T with $|T| \leq d$ and $|T| \neq |V| - 3$, creates exactly two strongly connected components one of them consists of a single vertex, the digraph Γ will be called *vosperian*. Clearly vosperian digraphs are superconnected. This notion was introduced by the authors of [12] as a step in their characterization of superconnected Abelian Cayley digraphs. As observed by Meng in [16], vosperianity is basically equivalent to the notion of hyper-connectdness, introduced independently in the undirected case, by Boesch [2].

A pair of vertices $\{x, y\}$ will be called a *twin pair* (resp. *anti-twin pair*) if x and y have the same successors (resp. predecessors). Following Meng [16], we shall say that a digraph having a twin pair or an anti-twin pair is *reducible*. The presence of twins or anti-twins is clearly an obstruction for vosperianity.

A recursive characterization of vosperian and superconnected Abelian Cayley digraphs was obtained by the authors of [12]. One of the authors [8] obtained a non-recursive characterization using the 2-atoms.

The description of vosperian and superconnected vertex-transitive digraphs has received considerable attention in recent years. Most of the characterizations use a result due to one of the authors in [8]. The case of edge-transitive undirected graphs was considered by Meng [16]. Liang, Meng and Zhang [13] described the case of bipartite undirected graphs. Arc transitive digraphs are investigated by Zhang and Meng in [17]. In the last paper it is shown that a strongly connected irreducible anti-symmetric arc-transitive digraph is either vosperian or bi-star superconnected. The last notion is avoidable in our approach, but the reader may refer to [17] for its definition. It is also shown that strongly connected irreducible anti-symmetric arc-transitive digraphs are superconnected.

More recently, the authors of [11] have given necessary and sufficient conditions for an undirected vertex-transitive graph to be vosperian. They have also proved that an irreducible superconnected vertex transitive undirected graph is vosperian [11].

We investigate in the present paper, the vosperianity and superconnectdness of vertex-transitive digraphs. We give a complete characterization of vosperian arc-transitive digraphs (Theorem 4) which completes the result of [17]. It states that an arc-transitive strongly connected digraph is vosperian if and only if it is irreducible. In particular, this is the case if the degree is coprime with the order of the digraph. We give also a complete characterization of vosperian Cayley digraphs (Theorem 6) and a complete characterization of irreducible superconnected Cayley digraphs (Theorem 7). These two last characterizations extend the corresponding ones in the directed case in [12, 8], and the ones for the undirected case in [11].

The paper is organized as follows. After giving some terminology and basic results in Section 2 we describe our main tools in Section 3. Section 4 is devoted to the study of the arc-transitive case. We then focus on Cayley digraphs and give the characterization of vosperianity in Section 5 and of superconnectedness for Cayley digraphs without twin pairs (i.e. defined by an aperiodic subset) in Section 6.

2 Terminology and preliminaries

Let $\Gamma = (V, E)$ be a digraph. The elements of V will be called *vertices*. The elements of E will be called *arcs*. Undirected graphs can be safely identified with symmetric digraphs. The *reverse* of Γ is the digraph Γ^- obtained by reversing the orientation of the arcs of E . More formally, $\Gamma^- = (V, E^-)$, where $E^- = \{(x, y) : (y, x) \in E\}$.

For a regular digraph Γ , we denote the degree of any vertex by $d(\Gamma)$. For a set $X \subset V$ we write $\Gamma(X) = \bigcup_{x \in X} \Gamma(x)$ and $\Gamma[X]$ will denote the subdigraph induced by X . A *source* of Γ is a subset $X \subset V$ such that $\Gamma^-(X) \subset X$. A *sink* of Γ is a subset $X \subset V$ such that $\Gamma(X) \subset X$. It is well known that a finite digraph is strongly connected if and only if it has no a proper subset which is a source (sink). As a warning we mention that our $\Gamma(X)$ is written $\Gamma^+(X)$ in some Graph Theory text books. Also our degree is sometimes called outdegree.

An *automorphism* of Γ is a bijection $f : V \rightarrow V$ such that $f(\Gamma(x)) = \Gamma(f(x))$, for every $x \in V$. A digraph is said to be *vertex-transitive* if, for any pair x, y of vertices, there is an automorphism that maps x into y . A digraph is said to be *arc-transitive* if, for any pair $(x, y), (x', y')$ of arcs, there is an automorphism that maps x into x' and maps y into y' . It is an easy exercise to show that a strongly connected arc-transitive digraph is also vertex-transitive, and hence regular.

Let G be a group and let $S \subset G \setminus \{1\}$. The digraph $\text{Cay}(G, S) = (G, E)$, where $E = \{(x, y) : x^{-1}y \in S\}$ is called a *Cayley* digraph. Recall that Γ is strongly connected if and only if S is a generating subset. Note that the left translation $\gamma_a : x \mapsto ax$ is an automorphism of $\text{Cay}(G, S)$. A set S of a group G is said to be *left-aperiodic* if for some $x \in G \setminus \{1\}$, $xS = S$. If both S and S^{-1} are not left-periodic, we shall say that S is *aperiodic*. The next lemma is just an exercise:

Lemma 1 *Let S be a subset of a group G . Then*

- (i) *$\text{Cay}(G, S) = (G, E)$ has a pair of twins if and only if S is left-periodic,*
- (ii) *$\text{Cay}(G, S) = (G, E)$ is irreducible if and only if S is aperiodic.*

Let $\Gamma = (V, E)$ be digraph. Given a subset $X \subset V$, the *boundary* of X is defined as

$$\partial_\Gamma(X) = \Gamma(X) \setminus X.$$

The *exterior* of X is defined as $\nabla_\Gamma(X) = V \setminus (X \cup \Gamma(X))$. We write $\partial_\Gamma^-(X) = \partial_{\Gamma^-}(X)$ and $\nabla_\Gamma^-(X) = \nabla_{\Gamma^-}(X)$. If the context is clear the reference to Γ will be omitted. Every set X induces the partition (with possibly empty parts) $\{X, \partial(X), \nabla(X)\}$ of the vertex set with no arc from X to $\nabla(X)$.

In particular,

$$\partial^-(\nabla(X)) \subset \partial(X). \tag{1}$$

The digraph Γ is said to be *k-separable* if there is a subset $X \subset V$ such that $\min\{|X|, |\nabla(X)|\} \geq k$. We shall say that a subset X of V induces a *k-separation* on Γ if $k \leq \min\{|X|, |\nabla(X)|\} < \infty$.

In particular, X induces a k -separation on Γ if and only if $\nabla(X)$ induces a k -separation on Γ^- . Hence, Γ is k -separable if and only if Γ^- is k -separable.

As an exercise, the reader may check that a subset $T \subset V$ is a cutset if and only if there is a nonempty subset $X \subset V$, with $\nabla(X) \neq \emptyset$ and $\partial(X) \subset T$. One may see from (1) that T is a cutset of Γ if and only if T is a cutset of Γ^- . A cutset of minimal cardinality, where $\min(\emptyset) = |V| - 1$, will be called a *minimum* cutset.

For a k -separable digraph Γ , the *kth-isoperimetric connectivity* of Γ is defined as

$$\kappa_k(\Gamma) = \min\{|\partial(X)| : X \subset V \text{ and } \min\{|X|, |\nabla(X)|\} \geq k\}.$$

A subset $F \subset V$ is called a *k-fragment* if $\min\{|F|, |\nabla(F)|\} \geq k$ and $|\partial(F)| = \kappa_k(\Gamma)$. A *k-atom* is a k -fragment of minimum cardinality. A k -fragment (a k -atom) of Γ^- is called a *negative k-fragment* (a *negative k-atom*). We also write $\kappa_{-k}(\Gamma) = \kappa_k(\Gamma^-)$.

The notion of *kth-isoperimetric connectivity* was introduced in [7]. We recall the following duality lemma:

Lemma A ([10]) *Let $\Gamma = (V, E)$ be a finite k -separable digraph. Then $\kappa_k(\Gamma) = \kappa_{-k}(\Gamma)$. If X is a k -fragment, then*

$$\partial^-(\nabla(X)) = \partial(X) \text{ and } \nabla^-(\nabla(X)) = X.$$

In particular, $\nabla(X)$ is a negative k -fragment.

It follows easily that $\kappa(\Gamma) = \kappa_1(\Gamma)$. We need a special case $k = 2$ of the easy next lemma:

Lemma B (folklore) *Let $\Gamma = (V, E)$ be a finite k -separable digraph and let A be a k -atom with $|A| > k$. Then $\Gamma^-(x) \cap A \neq \emptyset$, for every $x \in A$. In particular, A contains a circuit.*

3 Some tools

Let G be a group and let S be a subset of G . The subgroup generated by S will be denoted by $\langle S \rangle$. Let A, B be subsets of G . The *Minkowski product* is defined as

$$AB = \{xy : x \in A \text{ and } y \in B\}.$$

We use the following easy lemmas:

Lemma C ([15], **Theorem 1**) *Let G be a finite group and let A, B be subsets of G such that $|A| + |B| > |G|$. Then $AB = G$.*

Lemma D (folklore) *Let G be a finite cyclic group generated by r and let $B = \{1, r, \dots, r^{|B|-1}\}b$. For every subset $A \subset G$, such that $|AB| = |A| + |B| - 1 < |G|$, there is an $a \in G$ such that $aA = \{1, r, \dots, r^{|A|-1}\}$.*

Let $\Gamma = (V, E)$ be a finite digraph. A *block of imprimitivity* (or simply a block in what follows) of Γ is a subset $B \subset V$ such that for every automorphism f of Γ , either $f(B) = B$ or $f(B) \cap B = \emptyset$.

Recall the following result:

Lemma E (folklore) *Let G be a finite group and let S be a subset of G . Any block B of $\Gamma = \text{Cay}(G, S)$ with $1 \in B$ is a subgroup of G .*

As an easy exercise, the reader may show that a digraph Γ is vosperian if and only if it is either non 2-separable or if $\kappa_2(\Gamma) \geq d(\Gamma) + 1$. Thus, for finite digraphs Lemma A implies that Γ is non-vosperian if and only if Γ^- is non-vosperian.

We shall use some results from [9]. These results were originally formulated for directed graphs with loops. But they apply to our context, since the deletion (or addition) of loops does not modify the notions of k -fragments and k -isoperimetric connectivity. We use the following result:

Theorem F ([9]) *Let $\Gamma = (V, E)$ be a finite non-vosperian vertex-transitive digraph. Then one of the following holds:*

- (i) *There is a 2-atom of size 2 or a negative 2-atom of size 2.*
- (ii) *There is a block which is a 2-fragment or a negative 2-fragment.*
- (iii) *Every vertex of V is contained in at most two distinct 2-atoms. Moreover the intersection of two distinct 2-atoms has cardinality < 2 .*
- (iv) *Every vertex of V is contained in at most two distinct negative 2-atoms. Moreover the intersection of two distinct negative 2-atoms has cardinality < 2 .*

Note that the notion of superatoms, the smallest atoms with cardinality larger than one, is used in [9] instead of the close notion of 2-atoms used here.

Corollary 2 ([9]) *Let S be a generating subset of a finite group G with $1 \notin S$ such that $\Gamma = \text{Cay}(G, S)$ is non-vosperian. Then there are a subgroup H and an $a \in G$ such that $H \cup Ha$ is a 2-fragment or a negative 2-fragment.*

Proof. Let M be a 2-atom and let N be a negative 2-atom such that $1 \in M \cap N$. We may assume that $\min(|M|, |N|) \geq 3$. Otherwise the result holds with $H = \{1\}$. Moreover, Γ has no block which is a 2-fragment or a negative 2-fragment. Otherwise the result holds by Lemma E.

By Theorem F, 1 is contained in at most two 2-atoms of Γ or two 2-atoms of Γ^- . Up to replacing Γ by Γ^- , we may assume that 1 is contained in at most two 2-atoms of Γ . Let $H = \{x \in G : xM = M\}$. The result holds clearly if $M = H$. Take an $a \in M \setminus H$. We shall

show that $M = H \cup Ha$. Observe that M and $a^{-1}M$ are two distinct 2-atoms containing 1. Thus for every $x \in M$, we have $x^{-1}M = M$ or $x^{-1}M = a^{-1}M$. Therefore either $x \in H$ or $xa^{-1} \in H$. In particular, $M \subset H \cup Ha$. But $HM \subset M$, by the definition of H . \square

Let G be an Abelian group and let $1 \in A$ is a 2-atom of $\text{Cay}(G, S)$. It was proved in [8], that A is a subgroup if $|A| \geq 3$. An example given in [11] shows that this conclusion may fail in the non-abelian case.

4 Arc-transitive digraphs

Let $a \in V$. The *twin class* of a is $W_a = \{x : \Gamma(a) = \Gamma(x)\}$. We need the following lemma.

Lemma 3 *Let $\Gamma = (V, E)$ be vertex-transitive digraph and let v be an element of V . Then $\{W_a : a \in V\}$ is a partition of V . Moreover $\Gamma^-(v) = \bigcup_{a \in \Gamma^-(v)} W_a$. Also $|W_v|$ divides both $|V|$ and $d(\Gamma)$. In particular, Γ is irreducible if $\gcd(|V|, d(\Gamma)) = 1$.*

Proof. Assume that there is a x with $x \in W_a \cap W_b$, for some $a, b \in V$. We have $\Gamma(x) = \Gamma(a) = \Gamma(b)$. In particular, $a \in W_b$ and hence $W_a = W_b$. It follows that $\{W_a : a \in V\}$ is a partition of V . In particular, $|W_v|$ divides $|V|$.

Assume that $x \in \Gamma^-(v)$ and let y be an element of W_x . We have $v \in \Gamma(x)$ and $\Gamma(y) = \Gamma(x)$. Thus $y \in \Gamma^-(v)$. It follows that $\{W_a : a \in \Gamma^-(v)\}$ is a partition of $\Gamma^-(v)$. Take an arbitrary $x \in V$ and an automorphism f such that $f(v) = x$. Clearly $f(W_v) = W_x$. It follows that $|W_v| = |W_x|$, for every $x \in V$. Therefore, $|W_v|$ divides both $|V|$ and $d(\Gamma)$. \square

The next result gives a good description for vosperian arc-transitive digraphs.

Theorem 4 *A finite strongly connected arc-transitive digraph $\Gamma = (V, E)$ of degree $d \leq |V| - 4$ with $d \notin \{1, 2, 4, 6\}$ is vosperian if and only if it is irreducible.*

Proof. As we observed in the Introduction a reducible digraph is non-vosperian. Assume that Γ is an irreducible arc-transitive digraph which is non-vosperian. We prove the following points:

(i) No block is a 2-fragment.

Suppose to the contrary that some block B is a 2-fragment. Take $a \in B$ and an arc (a, b) with $b \notin B$. The set B must be an independent set, otherwise there would exist an arc (c, d) inside B . Since Γ is arc-transitive, there is an automorphism f with $f(c) = a$, and $f(d) = b$. Thus $B \cap f(B) \neq \emptyset$ but $B \neq f(B)$, contradicting the definition of a block. Hence, $\Gamma(x) \subset \partial(B)$, for every $x \in B$. Since $|\Gamma(x)| \leq |\partial(B)| \leq \kappa_2(\Gamma) \leq d$, we have $\Gamma(x) = \partial(B)$, for every $x \in B$. In particular Γ has a twin pair, a contradiction.

(ii) No block is a negative 2-fragment. The proof is similar to the proof of (i).

(iii) Every vertex of V is contained in at least three distinct 2-atoms or there are distinct 2-atoms M, N with $|M \cap N| \geq 2$.

Suppose on the contrary, that two distinct 2-atoms intersect in at most one point, so that every pair of vertices is contained in at most one 2-atom, and that every vertex is in at most two distinct 2-atoms.

Take a 2-atom A . Suppose that $|A| = 2$. Since $|\partial A| = \kappa_2(A) \leq d$ (otherwise Γ is vosperian) and the digraph has no twin pairs, $\Gamma[A]$ contains an arc (x, y) . Since the degree $d \geq 3$, the automorphisms sending (x, y) to (x, y') for each $y' \in \Gamma(x)$ provide at least three distinct 2-atoms containing x . Hence we may assume that $|A| > 2$.

Let us show that $\Gamma[A]$ is a strongly connected vertex transitive digraph. By Lemma B, $\Gamma[A]$ contains a circuit, say $C = [a_1, a_2, \dots, a_j]$. Take an arbitrary vertex $v \in A$. By Lemma B, there is $w \in A \cap \Gamma^-(v)$. Take an automorphism f with $f(a_1, a_2) = (w, v)$. Since every pair of vertices is contained in a unique atom, we have $f(A) = A$. Therefore $f(C)$ is a circuit in $\Gamma[A]$ containing v . Thus every arc of $\Gamma[A]$ is contained in a circuit of length at least two. If $\Gamma[A]$ is not strongly connected, it contains a strongly connected component K which is a sink, which must have cardinality at least two. We have $\partial(K) \subset \partial(A)$. Hence, by the minimality of a 2-atom $K = A$. It follows that $\Gamma[A]$ is strongly connected. Moreover, the same argument shows that the arbitrary vertex v can be sent to a_2 by an automorphism which leaves $\Gamma[A]$ invariant, so that this induced subgraph is vertex transitive and hence a regular digraph. We shall denote the degree of $\Gamma[A]$ by r .

There is an element $c \in A$ with an element $c' \in \Gamma(c) \cap \partial(A)$. By Lemma B, there is an arc (b, c) contained in A . For every arc $(x, y) \in E$, there is an automorphism f with $f(b) = x$ and $f(c) = y$. In particular $f(A)$ is a 2-atom containing $\{x, y\}$. By our hypothesis, such a 2-atom is unique. We shall denote this atom by A_{xy} . Since c is contained in at most two distinct 2-atoms, $\Gamma(c) = (\Gamma(c) \cap A) \cup (\Gamma(c) \cap A_{cc'})$. In particular, $d = 2r$. Let a be an element in $A \setminus \{b, c\}$ and let a', b' be vertices in the boundary of A such that $a' \in \Gamma(a)$ and $b' \in \Gamma(b)$. We have $\partial(A) \supset (\Gamma(a) \cap A_{aa'}) \cup (\Gamma(b) \cap A_{bb'}) \cup (\Gamma(c) \cap A_{cc'})$. Thus we have using our hypothesis on atoms intersection,

$$\begin{aligned} |\partial(A)| &\geq |\Gamma(a) \cap A_{aa'}| + |(\Gamma(b) \cap A_{bb'}) \setminus A_{aa'}| + |(\Gamma(c) \cap A_{cc'}) \setminus (A_{aa'} \cup A_{bb'})| \\ &\geq |\Gamma(a) \cap A_{aa'}| + |(\Gamma(b) \cap A_{bb'})| - 1 + |\Gamma(c) \cap A_{cc'}| - 2 \\ &= r + r - 1 + r - 2 = 3d/2 - 3 > d, \end{aligned}$$

since $d > 6$, this implies a contradiction.

(iv) Every vertex of V is contained in at least three distinct negative 2-atoms or there are distinct negative 2-atoms M, N with $|M \cap N| \geq 2$. The proof is exactly the same as for (iii).

By Theorem F, Γ has a 2-atom with size 2 or a negative 2-atom with size 2. The two cases are similar, we choose the first one. Since Γ has no twins, the 2-atom has the form $A = \{a, b\}$, with $(a, b) \in E$.

Let v be the only element of $\Gamma(b) \setminus \Gamma(a)$ and take $u \in \Gamma(b) \cap \Gamma(a)$. Now $\{b, v\}$ is a 2-atom. It follows that $u \in \Gamma(v)$. Also $\{b, u\}$ is a 2-atom. It follows that $v \in \Gamma(u)$, a contradiction in the anti-symmetric case. Assume now that Γ is symmetric. A similar reasoning proves that, $u \in \Gamma(z)$ where z is the only element of $\Gamma(a) \setminus \Gamma(b)$. Since $\{a, u\}$ is a 2-atoms, it follows that $|\Gamma(u) \cap (\Gamma(b) \cap \Gamma(a))| \geq d - 4$. Thus, $|\Gamma(u) \cap \Gamma(\{a, b\})| = d$ and hence, $\Gamma(u) \cap \nabla(\{a, b\}) = \emptyset$, a contradiction. \square

Corollary 5 *Let Γ be a strongly connected arc-transitive digraph of degree $d \notin \{2, 4, 6\}$ such that $\gcd(|V|, d) = 1$. Then Γ is vosperian.*

Proof. By Lemma 3, Γ is irreducible. The result follows now by Theorem 4. \square

5 Vosperian Cayley digraphs

In this section, we investigate vosperian Cayley digraphs.

Let X be a subset of a group G . We shall write $\tilde{X} = X \cup \{1\}$. We shall say that X is a *right r -coprogression*, if $G \setminus X = \{a, ra, \dots, r^j a\}$, for some $a \in G$. Similarly we define a *left r -coprogression*. Notice that S is a right r -coprogression if and only if S is a left $a^{-1}ra$ -coprogression.

In particular, if \tilde{S} is a right r -coprogression and $|S| \leq |G| - 4$ then

$$|\Gamma(\{1, r\}) \cup \{1, r\}| = |\{1, r, \} \tilde{S}| = |S| + 2, \quad (2)$$

and $\kappa_2(\Gamma) \leq |S|$.

Theorem 6 *Let S be a generating subset of a finite group G , with $1 \notin S$. Then $\Gamma = \text{Cay}(G, S)$ is non-vosperian if and only if one of the following holds:*

- (i) *There are a subgroup H of G with $|H| \geq 2$, and an element $a \in G$ such that $|(H \cup Ha)\tilde{S}| \leq \min(|G| - 2, |H \cup Ha| + |S|)$.*
- (ii) *There are a subgroup H of G with $|H| \geq 2$, and an element $a \in G$ such that $|\tilde{S}(H \cup aH)| \leq \min(|G| - 2, |H \cup Ha| + |S|)$.*
- (iii) *There is a $r \in G \setminus \{1\}$ such that \tilde{S} is a right r -coprogression and $|G| - 4 \geq |S|$.*

Proof. Let us first prove the sufficiency. If there exist a subgroup H with $|H| \geq 2$ and an element $a \in G$ such that (i) or (ii) holds, then Γ is 2-separable and $\kappa_2(\Gamma) \leq |S|$. Similarly, if there exists a $r \in G \setminus \{1\}$, such that \tilde{S} is a right r -coprogression, then $|\Gamma(\{1, r\}) \cup \{1, r\}| = |\{1, r, \} \tilde{S}| = |S| + 2$, and hence Γ is 2-separable and $\kappa_2(\Gamma) \leq |S|$. So each of these conditions is a necessary one.

Suppose now that Γ is non-vosperian. By Corollary 2, there are a subgroup H and an element $a \in G$ such that $H \cup Ha$ is a 2-fragment or a negative 2-fragment. The two cases are similar (and equivalent up to duality), so we shall consider only the case where $H \cup Ha$ is a 2-fragment. Assume first that $|H| \geq 2$. Since $H \cup Ha$ is a 2-fragment, we have $|(H \cup Ha)\tilde{S}| = |\Gamma(H \cup Ha)| \leq |G| - 2$. We have also $|H \cup Ha| + |S| \geq |H \cup Ha| + \kappa_2(\Gamma) = |(H \cup Ha)\tilde{S}|$. Therefore (i) holds.

Assume now that $|H| = 1$. Let $M = \{1, r\}$. By the definition of a 2-atom we have $|\{1, r\}\tilde{S}| = \kappa_2(\Gamma) + 2 \leq |S| + 2$. We may assume that $\kappa_2(\Gamma) = |S|$. Otherwise $r\tilde{S} = \tilde{S}$, and hence \tilde{S} is a union of right $\langle r \rangle$ -cosets, and thus Condition (i) holds with $H = \langle r \rangle$. Hence,

$$\kappa_2(\Gamma) = |S| \text{ and thus } |\{1, r\}\tilde{S}| = |S| + 2. \quad (3)$$

Let K be the cyclic subgroup generated by r . If $G = K$, then by (3), \tilde{S} is a progression and also a coprogression since r generates G . So we may assume that $K \neq G$.

Let $\tilde{S} = S_1 \cup \dots \cup S_j$ be the partition of \tilde{S} induced by the partition of G into right K -cosets. We have $j \geq 2$ since $K \neq G$. We shall assume that $|S_1| \leq \dots \leq |S_j|$. We must have $|S_2| = |K|$, since otherwise $|\{1, r\}S_i| \geq |S_i| + 1$, for all $1 \leq i \leq 2$. It would follow that $|\{1, r\}\tilde{S}| \geq |S_1| + 1 + |S_2| + 1 + \sum_{i \geq 3} |S_i| \geq |\tilde{S}| + 2$, contradicting (3). It follows also that $|\{1, r\}S_1| = |S_1| + 1$. Since r generates K , Lemma D implies that S_1 is a right progression with ratio r . Note that $KS_1 \setminus S_1$ is also a right progression.

Subcase 2.1 $j|K| = |G|$. In this case $G \setminus \tilde{S} = KS_1 \setminus S_1$ is also a right progression.

Subcase 2.2 $j|K| \leq |G| - |K|$. We have $(j-1)|K| + |S_1| - 1 = |\tilde{S}| - 1 = \kappa_2(\Gamma) \leq |K\tilde{S}| - |K| = (j-1)|K|$. Thus $|S_1| = 1$. In particular, K is clearly a 2-fragment and (i) holds with $H = K$ and $a = 1$. \square

6 Superconnected Cayley digraphs

In this section we characterize irreducible superconnected Cayley digraphs. As we have seen, these Cayley digraphs are defined by an aperiodic subsets.

Theorem 7 *An irreducible strongly connected Cayley digraph $\Gamma = \text{Cay}(G, S)$ on a finite group G is superconnected if and only if one of the following conditions holds:*

- (i) Γ is vosperian,
- (ii) for some $r \in G$, \tilde{S} is a right r -coprogression with $r^{-1} \notin S$ and $|G| - 4 \geq |S|$.

Proof. Take a subset S of G such that $\Gamma = \text{Cay}(G, S)$. Since Γ is strongly connected, S must be a generating subset of G .

Condition (i) implies obviously that Γ is superconnected. Assume that (ii) holds and let $P = \langle r \rangle \cap \tilde{S}$. By Lemma 1, S is aperiodic, and thus $|P| \geq 2$. Note that, since \tilde{S} is a coprogression $\kappa(\Gamma) = |S|$. Take now any fragment F of Γ . By a left-translation we can assume that $1 \in F$. We must have $F \subset H$, since otherwise $|FH| \geq 2|H|$, and hence by Lemma C,

$$|F\tilde{S}| \geq |F(\tilde{S} \setminus H)| = |(F)H(\tilde{S} \setminus H)| = |(FH)(\tilde{S} \setminus H)| = |G|,$$

a contradiction. Now we have $|F| + |\tilde{S}| - 1 = |F\tilde{S}| = |G \setminus H| + |FP| = |\tilde{S}| - |P| + |FP|$. In particular, $|H| - 1 \geq |FP| = |F| + |P| - 1$. By Lemma D, F is a r -progression. By a left-translation, we may assume that $F = \{1, r, \dots, r^{|F|-1}\}$. Since $|H| - 1 \geq |FP|$, we have $\Gamma(r^{|F|-1}) = \partial(F)$. Thus, $\text{Cay}(G, S)$ is superconnected.

Let us now prove the necessity.

Suppose that (i) and (ii) do not hold. Since Γ is non-voesperian, by Theorem 6 we are in one of the following cases:

Case 1. There are a subgroup H and an element $a \in G$ with $|H| \geq 2$ such that $|G| - 2 \geq |(H \cup Ha)\tilde{S}| = |H \cup Ha| + |\tilde{S}| - 1$.

Put $T = (H \cup Ha)\tilde{S} \setminus (H \cup Ha)$. Clearly T is a minimum cutset with $HT = T$. We can not have $T = \Gamma(x)$, for some x , otherwise $HxS = xS$, and S would be right-periodic, a contradiction. Also, we can not have $T = \Gamma^-(y)$, for some y , otherwise $HyS^{-1} = yS^{-1}$, and S^{-1} would be right-periodic, a contradiction. Thus Γ is non superconnected.

Case 2. There are a subgroup H and an element $a \in G$ with $|H| \geq 2$ such that $|G| - 2 \geq |(H \cup Ha)\tilde{S}^{-1}| = |H \cup Ha| + |\tilde{S}| - 1$. This case is similar to the previous one.

Case 3. For some $r \in G$, \tilde{S} is a right r -coprogression and $|S| \leq |G| - 4$. Put $H = \langle r \rangle$ and $G \setminus \tilde{S} \subset Ha$ for some $a \in G$.

Subcase 3.1 $a \notin H$. Since \tilde{S} is a coprogression, we have $H \subset \tilde{S}$. Since $(a^{-1}Ha) \cap Ha = \emptyset$, we have $a^{-1}Ha \subset \tilde{S}$. Take an arbitrary $h \in H \setminus \{1\}$. We have

$$\Gamma(ha) = haS \supset ha(a^{-1}Ha \setminus \{1\}) = Ha \setminus \{ha\}.$$

It follows that $\Gamma[Ha]$ is a complete symmetric digraph. Since $H \subset \tilde{S}$, we have $r, r^{-1} \in S$. Thus, $\Gamma[\{1, r\}]$ is a complete symmetric digraph. Hence $V \setminus \partial(\{1, r\})$ has exactly two strongly connected components each of them has size ≥ 2 .

Let $T = \partial(\{1, r\})$. Since $|\{1, r\}\tilde{S}| = |S| + 2$, we have $|T| = |S|$. By the observation made above $T \neq \Gamma(x)$, for every $x \in G$. Therefore, Γ is non superconnected.

Subcase 3.2 $a \in H$. Since (ii) does not hold, we have $r, r^{-1} \in S$. Put $P = \langle r \rangle \cap \tilde{S}$. Let F be any fragment with $1 \in F$. We must have $F \subset H$, since otherwise by Lemma C,

$$|F\tilde{S}| \geq |F(\tilde{S} \setminus H)| = |(F)H(\tilde{S} \setminus H)| = |(FH)(\tilde{S} \setminus H)| = |G|,$$

a contradiction.

Now we have $|F| + |\tilde{S}| - 1 = |F\tilde{S}| = |G \setminus H| + |FP| = |\tilde{S}| - |P| + |FP|$. In particular, $|H| - 1 \geq |FP| = |F| + |P| - 1$. By Lemma D, F is a r -progression. Since $\{r, -r\} \subset S$, the set F is a strongly connected subset. Similarly $G \setminus \Gamma(F)$ is a strongly connected subset. Therefore Γ is non superconnected. \square

Corollary 8 *Let S be an aperiodic generating subset of a finite group G , with $1 \notin S$, and $|S| \leq |G|/2$. Then $\text{Cay}(G, S)$ is superconnected if and only if one of the following holds:*

- $\text{Cay}(G, S)$ is voesperian,
- There is a $r \in G$ such that G is a cyclic group and $S = \{r, r^2, \dots, r^{|S|}\}$.

Proof. Let us see that if $|S| \leq |G|/2$ holds then a coprogression is also a progression. If $G \setminus \tilde{S} \subset \langle r \rangle$ then $G = \langle r \rangle$. Since otherwise, $|\tilde{S}| \geq |G| - |\langle a \rangle| + 2 \geq \frac{|G|}{2} + 2$, a contradiction. The result follows now by Theorem 7. \square

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